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CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 49

Stock Price Modelling

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STOCK PRICE DISTRIBUTION OVER AN INFINITESIMAL TIME PERIOD

- Stock prices are assumed to follow a Markov process.
- The Markov property of stock prices is consistent with the weak form of market efficiency i.e. that the current market price encapsulates its entire past history.
- The current price moves only when the market receives any relevant new information.
- The past history of prices is irrelevant for future price prediction since it is captured in the current price.



- The expected percentage return per unit time required by investors over an infinitesimal time period dt from a stock is independent of the stock's price. If investors require a 15% per annum expected return when the stock price is 10, then, they will also require a 15% per annum expected return when it is 50.
- The variance of percentage change in stock price in time dt is independent of the stock price. An investor is just as uncertain of the percentage return when the stock price is 50 as when it is 10.



The expected percentage return per unit time required by investors from a stock in an infinitesimal time dt is constant and independent of the stock price i.e.

$$E(R) = E\left[\frac{1}{dt}\left(\frac{dS}{S_0}\right)\right] = \mu \text{ or } E\left(\frac{dS}{S_0}\right) = \mu dt$$

$$\frac{dS}{S_0} = \mu dt + \text{randomness (with zero mean)}$$

The variability of the percentage change in stock price in a short period of time dt is the same irrespective of the stock price so that

$$\text{Var}\left(\frac{dS}{S_0}\right) = \sigma^2 dt \quad \text{or} \quad \text{SD}\left(\frac{dS}{S_0}\right) = \sigma \sqrt{dt}$$

$$\frac{dS}{S_0} = \mu dt + \sigma dW = \mu dt + \sigma \sqrt{dt} Z$$

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$$dS = \mu S_0 dt + \sigma S_0 dW = \mu S_0 dt + \sigma S_0 \sqrt{dt} Z$$

$$dS = S_{dt} - S_0 \xrightarrow{\text{distribution}} N(\mu S_0 dt, \sigma^2 S_0^2 dt)$$

$$S_{dt} \xrightarrow{\text{distribution}} N(S_0 + \mu S_0 dt, \sigma^2 S_0^2 dt)$$

**LOG
NORMAL
PROPERTY
OF STOCK
PRICES**

From Ito's Lemma, function $G(\xi, t)$ of a stochastic variable ξ , satisfies the Ito equation

$$dG = \left(a \cdot \frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \cdot \frac{\partial^2 G}{\partial \xi^2} \right) dt + b \cdot \frac{\partial G}{\partial \xi} dW \quad (1)$$

$$\text{where } d\xi = a(\xi, t)dt + b(\xi, t)dW \quad (2).$$

From this eq. we see that in an infinitesimal time interval dt , the process G is stochastic with a drift rate

$$\left(a \cdot \frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \cdot \frac{\partial^2 G}{\partial \xi^2} \right), \text{variance rate } \left(b \cdot \frac{\partial G}{\partial \xi} \right)^2$$



Let $\xi = S$, $G(\xi, t) = \ln S$. where $dS = \mu S dt + \sigma S dW$.

Then, $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$

Also, in our problem, $dS = \mu S dt + \sigma S dW$

so that on comparing with

$d\xi = a(\xi, t)dt + b(\xi, t)dW$ we get $a = \mu S$ and $b = \sigma S$

Using Ito's lemma :

$$dG = \left(a \cdot \frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \cdot \frac{\partial^2 G}{\partial \xi^2} \right) dt + b \cdot \frac{\partial G}{\partial \xi} dW$$

$$d(\ln S) = \left(\mu S \cdot \frac{1}{S} + 0 - \frac{1}{2} \sigma^2 S^2 \cdot \frac{1}{S^2} \right) dt + \sigma S \cdot \frac{1}{S} dW$$

$$= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW$$

Thus, by Ito eq.

$$\mathbf{dG = d(\ln S) = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW.}$$

Thus, $G = \ln S$ follows a generalized Wiener process

with drift rate $\left(\mu - \frac{\sigma^2}{2} \right)$ and variance rate σ^2 .

Integrating : $\ln S_T - \ln S_0 = \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma W_T$

Thus, the increment $(\ln S_T - \ln S_0)$ is normally distributed with mean $\left(\mu - \frac{\sigma^2}{2} \right) T$ and variance $\sigma^2 T$

i.e. $\ln S_T - \ln S_0 \xrightarrow{\text{distribution}} N \left[\left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right]$

whence $\ln S_T \xrightarrow{\text{distribution}} N \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right]$

STOCK PRICE MODEL: ALTERNATIVE TREATMENT

Langevin SDE : $dS(t) = \mu S(t)dt + \sigma S(t)dW$ or $\frac{1}{dt} \left(\frac{dS}{S} \right) = \mu + \sigma \eta(t)$

Fokker Planck Equation for transition probabilities :

$$\frac{\partial}{\partial t} p(S(t), t | S_0, 0) = -\frac{\partial}{\partial S} [\mu S(t) p(S(t), t | S_0, 0)] + \frac{1}{2} \frac{\partial^2}{\partial S^2} [\sigma^2 S^2(t) p(S(t), t | S_0, 0)]$$

$$\text{Solution : } p(S_T, T | S_0, 0) = \frac{1}{S_T \sqrt{2\pi\sigma^2 T}} \times \exp \left\{ -\frac{\ln S_T - \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T \right]^2}{2\sigma^2 T} \right\}$$

PDF OF LOGNORMAL DISTRIBUTION

Let X be lognormally distributed with pdf $\rho(x)$.

Then $Y = \ln X$ is normally distributed.

Let $Y \xrightarrow{\text{distribution}} N(\lambda, \theta^2)$

$$p(y) = \frac{1}{\sqrt{2\pi\theta^2}} \exp\left[-\frac{(y-\lambda)^2}{2\theta^2}\right]$$

$$\begin{aligned} P(x < X < x + dx) &= \rho(x) dx = P(\ln x < \ln X < \ln(x + dx)) \\ &= P(y < Y < y + dy) = p(y) dy = p(y) [(y + dy) - y] = p(\ln x) [\ln(x + dx) - \ln x] \\ &= p(\ln x) \left[\ln\left(1 + \frac{dx}{x}\right) \right] = p(\ln x) \frac{dx}{x} = \frac{dx}{x\sqrt{2\pi\theta^2}} \exp\left[-\frac{(\ln x - \lambda)^2}{2\theta^2}\right] \end{aligned}$$

INTERESTING OBSERVATION

$$\mathbf{E} \left(S_T \right) = S_0 \mathbf{exp} \left(\mu T \right)$$

$$\mathbf{lnE} \left(S_T \right) = \mathbf{lnS}_0 + \left(\mu T \right)$$

$$\mathbf{Also} \quad \mathbf{E} \left(\mathbf{ln} \ S_T \right) = \mathbf{ln} \ S_0 + \left(\mu - \frac{1}{2} \sigma^2 \right) T$$

$$\mathbf{so \ that} \ \mathbf{lnE} \left(S_T \right) \neq \mathbf{E} \left(\mathbf{ln} \ S_T \right)$$

We define the logarithmic return as

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}. \text{ Since,}$$

$$\ln S_T \xrightarrow{\text{distribution}} N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$$

$$\text{it follows that } x = \frac{1}{T} \ln \frac{S_T}{S_0} \xrightarrow{\text{distribution}} N\left(\left(\mu - \frac{1}{2}\sigma^2\right), \frac{\sigma^2}{T}\right)$$

We define the logarithmic return in dt as

$$\begin{aligned}x &= \frac{1}{dt} \ln \frac{S_{dt}}{S_0} = \frac{1}{dt} \ln \left[1 + \frac{S_{dt} - S_0}{S_0} \right] = \frac{1}{dt} \ln \left[1 + \frac{dS}{S_0} \right] \\&= \frac{1}{dt} \left[\left(\frac{dS}{S_0} \right) - \frac{1}{2} \left(\frac{dS}{S_0} \right)^2 + \dots \right] \sim \frac{1}{dt} \left(\frac{dS}{S_0} \right) = \text{AR to first order.}\end{aligned}$$

Thus, $\text{AR} = \frac{1}{dt} \left(\frac{dS}{S_0} \right)$ is simply a first order approx.

of log returns and holds for small values of $\left(\frac{dS}{S_0} \right)$.